

Characterization of cyclic Schur groups

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Abstract

A finite group G is called a Schur group, if any Schur ring over G is associated in a natural way with a subgroup of $\text{Sym}(G)$ that contains all right translations. It was proved by R. Pöschel (1974) that given a prime $p \geq 5$ a p -group is Schur if and only if it is cyclic. We prove that a cyclic group of order n is Schur if and only if n belongs to one of the following five families of integers: p^k , pq^k , $2pq^k$, pqr , $2pqr$ where p, q, r are distinct primes, and $k \geq 0$ is an integer.

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1 Introduction

Let G be a finite group. A subring of the group ring $\mathbb{Q}G$ is called a *Schur ring* or *S-ring* over G , if it is closed with respect to the componentwise multiplication and inversion. The first construction of such a ring was proposed by I. Schur [8] in connection with his famous result on permutation groups containing a regular cyclic subgroup. Namely, let Γ be a permutation group on the set G that contains the regular group G_{right} induced by right multiplications,

$$G_{right} \leq \Gamma \leq \text{Sym}(G).$$

Denote by Γ_1 the stabilizer of the identity of G in Γ . Then the submodule of $\mathbb{Q}G$ spanned by the Γ_1 -orbits (transitivity module) is an S-ring over G . Such an S-ring was called *schurian* in [7]. The general theory of S-rings was developed by H. Wielandt in [9] where in particular he constructed an S-ring which cannot be obtained by the Schur method.

Definition (R. Pöschel). *A finite group G is called Schur, if any S-ring over G is schurian.*

The Wielandt example shows that not every finite group is Schur. More exactly, he proved that the group $\mathbb{Z}_p \times \mathbb{Z}_p$ is not Schur for prime $p \geq 5$. This fact was used by R. Pöschel in [7] to prove the following theorem.

Theorem. *Any section of a Schur group is a Schur group. Moreover, for a prime $p \geq 5$ a p -group is Schur if and only if it is cyclic.■*

Since any finite nilpotent group is a direct product of its Sylow subgroups, we immediately obtain the following result.

Corollary. *A nilpotent group of order coprime to 6 is Schur only if it is cyclic.■*

The above results show the importance of the cyclic case for the characterization of Schur groups. It should be noted that by the Pöschel theorem any cyclic p -group is Schur for $p \geq 5$. In fact, the schurity of cyclic 3-groups was also proved in [7], whereas the same result for $p = 2$ was obtained in [5]. However, till 2001 no cyclic non-Schur group was known, and moreover it was conjectured that all cyclic groups are Schur (the Schur-Klin conjecture). This conjecture had also been supported by the fact that the group \mathbb{Z}_n where n is a product of two distinct primes, is a Schur one [4]. The first counterexamples to the conjecture were constructed in [1]; in all these examples n was

the product of at least four primes. Later in [3] the schurity of \mathbb{Z}_n was proved when n is the product of at most three primes or $n = p^3q$ where p and q are distinct primes. The main result of this paper completes the characterization of cyclic Schur groups.

Theorem 1.1 *A cyclic group of order n is Schur if and only if n belongs to one of the following five (partially overlapped) families of integers:*

$$p^k, pq^k, 2pq^k, pqr, 2pqr \quad (1)$$

where p, q, r are distinct primes, and $k \geq 0$ is an integer.

Corollary 1.2 *The minimum order of a cyclic non-Schur group equals 72.■*

Let us briefly outline the proof of Theorem 1.1. To prove the necessity, for each integer n satisfying the hypothesis of Theorem 2.1 we construct explicitly a non-schurian S-ring over a group \mathbb{Z}_n . This ring is the generalized wreath product of two smaller schurian S-rings each of which is in its turn the generalized wreath product of normal S-rings; the way is essentially the same as one used in [1]. It turns out (Lemma 2.2) that the complement to the set of all these n coincides with the set of all numbers listed in (1).

To prove the sufficiency we have to verify that any S-ring over a cyclic group of order n belonging to one of families (1), is schurian. We observe that any divisor of such n also belongs to at least one of these families.

Definition 1.3 *A non-schurian S-ring \mathcal{A} over a group G is called minimal if the S-ring \mathcal{A}_S is schurian for any \mathcal{A} -section $S \neq G/1$.*

It is easily seen that any non-schurian S-ring contains a section the restriction to which is minimal non-schurian. Thus the sufficiency in Theorem 1.1 immediately follows from the theorem below.

Theorem 1.4 *The order of the underlying group of a minimal non-schurian circulant S-ring cannot belong to any of families (1).*

There are two key observations to prove Theorem 1.4 that are based on the results of [3]. The first is that any non-schurian circulant S-ring is a fusion of a quasidense¹ non-schurian circulant S-ring (Theorem 3.4). The second

¹Quasidense circulant S-rings are introduced and studied in Section 3.

is that such an S-ring is a proper generalized wreath product of two smaller schurian quasidense S-rings. Moreover, in the minimal case each of them is in its turn a proper generalized wreath product (Theorem 4.3). We use these observations in Sections 4 and 5 to exclude the first, second and fourth families, and the case $p = 2$ in the other two families. The proof is completed in Section 6 by applying the criterion of schurity for S-rings of special form that was proved in Section 8. It should be mentioned that throughout the proof of Theorem 1.4 we use several auxiliary results on circulant S-rings that are collected in Section 7.

In this paper we follow the notation and terminology of paper [3]. When referring to this paper we keep only the number of the statement, preceding it by the letter A (e.g. instead of [3, Theorem 4.1] we write Theorem A4.1). Several additional notations are listed below.

We write $\mathcal{A} \cong \mathcal{A}'$ when S-rings \mathcal{A} and \mathcal{A}' are Cayley isomorphic.

For an S-ring \mathcal{A} and an \mathcal{A} -section S we set $\text{Hol}_{\mathcal{A}}(S) = \text{Hol}(S) \cap \text{Aut}(\mathcal{A}_S)$.

For an S-ring \mathcal{A} over a group G we set

$$\mathcal{M}(\mathcal{A}) = \{\Gamma \leq \text{Aut}(\mathcal{A}) : \Gamma \approx_2 \text{Aut}(\mathcal{A}) \text{ and } G_{\text{right}} \leq \Gamma\}.$$

2 Necessity in Theorem 1.1

Here we prove the necessity of Theorem 1.1. Throughout this section \mathbb{Z}_n is the additive group of integers modulo a positive integer n .

For any divisor m of n denote by $i_{m,n} : \mathbb{Z}_m \rightarrow \mathbb{Z}_n$ and $\pi_{n,m} : \mathbb{Z}_n \rightarrow \mathbb{Z}_m$ the group homomorphisms taking 1 to n/m and to 1 respectively. Using them we identify the groups $i_{m,n}(\mathbb{Z}_m)$ and $\mathbb{Z}_n / \ker(\pi_{n,m})$ with \mathbb{Z}_m . Thus every section of \mathbb{Z}_n of order m is identified with the group \mathbb{Z}_m . Moreover, the permutation $f \in \text{Aut}(\mathbb{Z}_n)$ afforded by multiplication by an integer induces the permutation $f^m \in \text{Aut}(\mathbb{Z}_m)$ afforded by multiplication by the same integer.

If \mathcal{A} is an S-ring over $G = \mathbb{Z}_n$ and H is the \mathcal{A} -group of order m , then \mathcal{A}_H $\mathcal{A}_{G/H}$ are denoted respectively by \mathcal{A}_m and $\mathcal{A}^{n/m}$. Let finally \mathcal{A}_i be an S-ring over \mathbb{Z}_{n_i} ($i = 1, 2$) and $(\mathcal{A}_1)^m = (\mathcal{A}_2)_m$ for some m dividing both n_1 and n_2 . Then the unique S-ring \mathcal{A} over $\mathbb{Z}_{n_1 n_2 / m}$ from Theorem A3.4 is denoted by $\mathcal{A}_1 \wr_m \mathcal{A}_2$. We omit m if $m = 1$.

Below given a positive integer m we set

$$\Omega^*(m) = \begin{cases} \Omega(m), & \text{if } m \text{ is odd,} \\ \Omega(m/2), & \text{if } m \text{ is even} \end{cases}$$

where $\Omega(m)$ is the total number of prime factors of m . We observe that $\Omega(m) \leq 1$ if and only if m is a divisor of twice a prime number.

Theorem 2.1 *Let $n = n_1 n_2$ where n_1 and n_2 are coprime positive integers such that $\Omega^*(n_i) \geq 2$, $i = 1, 2$. Then a cyclic group of order n is not Schur.*

Proof. Below for an integer $m \geq 3$ we denote by K_m the subgroup of order 2 in the group $\text{Aut}(\mathbb{Z}_m)$ that is generated by multiplication by -1 . Suppose first that $n_1 = ab$ and $n_2 = cd$ where $a, b, c, d \geq 3$ are integers. Set

$$\mathcal{A}_1 = \text{Cyc}(K_a \times K_c, \mathbb{Z}_{ac}), \quad \mathcal{A}_2 = \text{Cyc}(K_{bc}, \mathbb{Z}_{bc}), \quad (2)$$

$$\mathcal{A}_3 = \text{Cyc}(K_{ad}, \mathbb{Z}_{ad}), \quad \mathcal{A}_4 = \text{Cyc}(K_{bd}, \mathbb{Z}_{bd}). \quad (3)$$

It is easily seen that the group $\text{Aut}(\mathcal{A}_i)$ is dihedral for $i = 2, 3, 4$, and is the direct product of two dihedral groups for $i = 1$. Therefore the S-ring \mathcal{A}_i is normal for all i . Moreover, $(\mathcal{A}_1)^c = \text{Cyc}(K_c, \mathbb{Z}_c) = (\mathcal{A}_2)_c$ and $(\mathcal{A}_3)^d = \text{Cyc}(K_d, \mathbb{Z}_d) = (\mathcal{A}_4)_d$. Thus one can form S-rings

$$\mathcal{A}_{1,2} = \mathcal{A}_1 \wr_c \mathcal{A}_2 \quad \text{and} \quad \mathcal{A}_{3,4} = \mathcal{A}_3 \wr_d \mathcal{A}_4.$$

It is easily seen that $(\mathcal{A}_{1,2})^{n_1} = \text{Cyc}(K_a, \mathbb{Z}_a) \wr \text{Cyc}(K_b, \mathbb{Z}_b) = (\mathcal{A}_{3,4})_{n_1}$. Then

$$\mathcal{A} := \mathcal{A}_{1,2} \wr_{n_1} \mathcal{A}_{3,4}$$

is an S-ring over \mathbb{Z}_n . Thus it suffices to verify that \mathcal{A} is not schurian.

Suppose on the contrary that \mathcal{A} is schurian. Then by Theorem A1.2 the S-rings $\mathcal{A}_{1,2}$ and $\mathcal{A}_{3,4}$ are schurian, and there exist groups $\Delta_{1,2} \in \mathcal{M}(\mathcal{A}_{1,2})$ and $\Delta_{3,4} \in \mathcal{M}(\mathcal{A}_{3,4})$ such that

$$(\Delta_{1,2})^S = (\Delta_{3,4})^S$$

where S is the section of order n_1 used in the definition of the S-ring \mathcal{A} . In particular, for any permutation $f_1 \in \Delta_{1,2}$ fixing 0 there exists a permutation $f_2 \in \Delta_{3,4}$ fixing 0 and such that $f_1^S = f_2^S$. We claim: *the permutation $(f_1)^H$*

where H is the group of order ac , is induced by multiplication by $\varepsilon \in \{1, -1\}$. However, if this is true, then the stabilizer of 0 in the group $(\Delta_{1,2})^H$ is contained in K_{ac} . Therefore the basic set of the S-ring associated with the former group that contains 1 is of cardinality ≤ 2 . On the other hand, this S-ring coincides with \mathcal{A}_1 by the schurity of the S-ring $\mathcal{A}_{1,2}$ and the 2-equivalence of the groups $\Delta_{1,2}$ and $\text{Aut}(\mathcal{A}_{1,2})$. So the above basic set has cardinality 4. Contradiction.

To prove the claim let $f_{1,1}$ and $f_{1,2}$ be the automorphisms of the S-rings \mathcal{A}_1 and \mathcal{A}_2 induced by f_1 , and $f_{2,3}$ and $f_{2,4}$ the automorphisms of the S-rings \mathcal{A}_3 and \mathcal{A}_4 induced by f_2 . Then the normality of these S-rings implies that $f_{1,1} \in K_a \times K_c$, $f_{1,2} \in K_{bc}$, and that $f_{2,3} \in K_{ad}$ and $f_{2,4} \in K_{bd}$. Clearly,

$$(f_{1,1})^c = (f_{1,2})^c \quad \text{and} \quad (f_{2,3})^d = (f_{2,4})^d \quad (4)$$

and due to the equality $(f_1)^S = (f_2)^S$ also

$$(f_{1,1})^a = (f_{2,3})^a \quad \text{and} \quad (f_{1,2})^b = (f_{2,4})^b. \quad (5)$$

Next, the permutations $f_{1,2}$, $f_{2,3}$ and $f_{2,4}$ are induced respectively by multiplications by some integers $\varepsilon_{1,2}, \varepsilon_{2,3}, \varepsilon_{2,4} \in \{1, -1\}$. Therefore, by the second equalities of (4) and (5) we have

$$\varepsilon_{1,2} = \varepsilon_{2,3} = \varepsilon_{2,4}.$$

Denote this number by ε . Then by the first equalities of (4) and (5) the permutations $(f_{1,1})^a$ and $(f_{1,1})^c$, and hence the permutation $(f_1)^H$, are induced by multiplication by ε .

To complete the proof we observe that the theorem is proved in all cases except for the case when one of the numbers n_1, n_2 , say n_1 , is equal to 8. Then obviously $n_1 = ab/2$ and $n_2 = cd$ where $a = b = 4$ and $c, d \geq 3$ are odd integers. Let us define S-rings $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 by formulas (2) and (3). Then again all these rings are normal,

$$(\mathcal{A}_1)^{2c} = \text{Cyc}(K_{2c}, \mathbb{Z}_{2c}) = (\mathcal{A}_2)_{2c}, \quad (\mathcal{A}_3)^{2d} = \text{Cyc}(K_{2d}, \mathbb{Z}_{2d}) = (\mathcal{A}_4)_{2d},$$

and one can form S-rings

$$\mathcal{A}_{1,2} = \mathcal{A}_1 \wr_{2c} \mathcal{A}_2 \quad \text{and} \quad \mathcal{A}_{3,4} = \mathcal{A}_3 \wr_{2d} \mathcal{A}_4.$$

It should be stressed that $\mathcal{A}_{1,2}$ and $\mathcal{A}_{3,4}$ are S-rings over the groups \mathbb{Z}_{cn_1} and \mathbb{Z}_{dn_1} . It is also easily seen that

$$(\mathcal{A}_{1,2})^{n_1} = \text{Cyc}(K_a, \mathbb{Z}_a) \wr_2 \text{Cyc}(K_b, \mathbb{Z}_b) = (\mathcal{A}_{3,4})_{n_1}.$$

Then

$$\mathcal{A} := \mathcal{A}_{1,2} \wr_{n_1} \mathcal{A}_{3,4}$$

is an S-ring over \mathbb{Z}_n . Thus it suffices to verify that \mathcal{A} is not schurian. The rest of the proof repeats the proof of the first part literally. ■

To complete the proof of the necessity we note that the required statement immediately follows from Theorem 2.1 and the lemma below.

Lemma 2.2 *An integer n belongs to none of the families listed in (1) if and only if $n = n_1 n_2$ for some coprime positive integers n_1 and n_2 such that $\Omega^*(n_i) \geq 2$, $i = 1, 2$.*

Proof. The sufficiency is straightforward by exhaustive search. To prove the necessity let an integer $n = p_1^{k_1} \cdots p_s^{k_s}$ belong to none of families (1) where p_1, \dots, p_s are pairwise distinct primes. Then without loss of generality we can assume that

$$2 \leq s \leq 4 \quad \text{and} \quad k_1 \geq k_2 \geq \cdots \geq k_s.$$

Suppose on the contrary that n cannot be decomposed into the product of coprime positive integers n_1 and n_2 such that $\Omega^*(n_i) \geq 2$, $i = 1, 2$. Then $k_2 = 1$, for otherwise $s = 2$ and n belongs to the third family with $p = 2$, which is impossible. Thus $k_2 = \cdots = k_s = 1$. Therefore $s = 3$ or $s = 4$, for otherwise $s = 2$ and n belongs to the second family. Let $s = 3$. Then $k_1 \neq 1$ because n does not belong to the fourth family. So $k_1 \geq 2$, and hence $2 \in \{p_2, p_3\}$ by the supposition. However, then n belongs to the third family. Contradiction. Finally, let $s = 4$. Then the supposition implies that $k_1 = 1$ and one of the p_i 's equals 2. But then n belongs to the fifth family. Contradiction. ■

3 Quasidense S-rings

A circulant S-ring \mathcal{A} is called *quasidense*, if any primitive \mathcal{A} -section is of prime order. Any dense S-ring is obviously quasidense. It is also clear that

the property to be quasidense is preserved by taking the restriction to any \mathcal{A} -section. Moreover, in the quasidense case any minimal \mathcal{A} -group is of prime order, any maximal \mathcal{A} -group is of prime index, and the S-ring \mathcal{A}_S is dense for any \mathcal{A} -section S of prime power order.

Theorem 3.1 *Any quasidense circulant S-ring with trivial radical is cyclotomic, and hence dense.*

Proof. Let \mathcal{A} be a quasidense circulant S-ring with trivial radical. Then \mathcal{A} is the tensor product of a normal S-ring with trivial radical and S-rings of rank 2 by Theorem A4.1. However, any normal circulant S-ring is cyclotomic by Theorem A4.2. Besides, by the quasidensity the underlying group of any factor of rank 2 is of prime order. Therefore such a factor is also cyclotomic. Thus \mathcal{A} is cyclotomic as the tensor product of cyclotomic S-rings. ■

The following two statements will be used in proving the sufficiency of Theorem 1.1 to find nontrivial \mathcal{A} -groups.

Corollary 3.2 *Let \mathcal{A} be a quasidense S-ring over a cyclic group G . Then any subgroup of G that contains $\text{rad}(\mathcal{A})$ is an \mathcal{A} -group. In particular, if $\text{rad}(\mathcal{A})_p = 1$ for a prime divisor p of $|G|$, then $G_{p'}$ is an \mathcal{A} -group.*

Proof. The S-ring $\mathcal{A}_{G/L}$ where $L = \text{rad}(\mathcal{A})$, has trivial radical. Therefore by Theorem 3.1 it is dense. Thus required statement follows from the fact that a group H containing L is an \mathcal{A} -group if and only if the group H/L is an $\mathcal{A}_{G/L}$ -group. ■

Corollary 3.3 *Let \mathcal{A} be a quasidense S-ring over a cyclic group G . Suppose that \mathcal{A} is not the U/L -wreath product where U/L is an \mathcal{A} -section such that the number $p := |L|$ is prime. Then*

- (1) *there exists $H \in \mathcal{G}(\mathcal{A})$ such that $H \not\leq U$ and $H_{p'} \in \mathcal{G}(\mathcal{A})$,*
- (2) *if $q := |G/U|$ is a prime other than p , then $H_{p'} \geq G_q$ for any group H from statement (1).*

Proof. To prove statement (1) we observe that by the hypothesis there exists $X \in \mathcal{S}(\mathcal{A})$ outside U such that $\text{rad}(X)_p = 1$. Then $H = \langle X \rangle$ is an \mathcal{A} -group. Therefore the required statement follows from Corollary 3.2 applied to the S-ring \mathcal{A}_H . Next, the condition of statement (2) implies that any group $H \not\leq U$ contains a generator of G_q . Therefore $H \geq G_q$ which proves this statement. ■

The following theorem reduces the schurity problem for circulant S-rings to the quasidense case. The proof is based on the extension construction studied in [3].

Theorem 3.4 *Given a circulant S-ring \mathcal{A} there exists a quasidense S-ring $\mathcal{A}' \geq \mathcal{A}$ such that \mathcal{A} and \mathcal{A}' are schurian or not simultaneously.*

Proof. Let us define an S-ring \mathcal{A}' recursively as follows. If \mathcal{A} has no singular class of composite order, then we set $\mathcal{A}' = \mathcal{A}$; otherwise we set

$$\mathcal{A}' = (\text{Ext}_C(\mathcal{A}, \mathbb{Z}S))'$$

where C is a singular class of composite order and $S = S_{\min}(C)$. Then the S-ring \mathcal{A}' has no singular classes of composite order. Moreover, from Theorem A6.7 it follows that \mathcal{A} and \mathcal{A}' are schurian or not simultaneously. To complete the proof let us verify that the S-ring \mathcal{A}' is quasidense. Suppose on the contrary that this is not true. Then there exists a primitive \mathcal{A}' -section S of composite order. Then by Theorem A4.6 the class of projectively equivalent \mathcal{A}' -sections that contains S , is singular. Contradiction. ■

In general, the automorphism group of a quasidense S-ring is not solvable. However, from Theorem A8.1 it follows that in the schurian case such an S-ring can always be obtained from an appropriate solvable permutation group in a standard way. The following theorem shows that “locally” this group has a rather simple form.

Theorem 3.5 *Let \mathcal{A} be a schurian quasidense circulant S-ring. Then there exists a group $\Gamma \in \mathcal{M}(\mathcal{A})$ such that $\Gamma^S = \text{Hol}_{\mathcal{A}}(S)$ for any \mathcal{A} -section S with $\text{rad}(\mathcal{A}_S) = 1$.*

Remark 3.6 *In fact, we prove that the equality in the theorem statement holds for any S such that \mathcal{A}_S is the tensor product of a normal S-ring and S-rings of rank 2.*

Proof. The quasidensity of \mathcal{A} implies that each primitive \mathcal{A} -section is of prime order. Therefore by Theorems A4.6 and A8.1 there exists a group $\Gamma \in \mathcal{M}(\mathcal{A})$ such that $\Gamma^T \leq \text{Hol}(T)$ for all primitive \mathcal{A} -sections T . Let S be an \mathcal{A} -section with $\text{rad}(\mathcal{A}_S) = 1$. Then by Theorem A4.1 the S-ring \mathcal{A}_S is

the tensor product of a normal S-ring, say \mathcal{A}_{T_0} , and S-rings of rank 2, say $\mathcal{A}_{T_1}, \dots, \mathcal{A}_{T_k}$ where T_i 's are \mathcal{A}_S -groups. It follows that

$$\Gamma^{T_0} \leq \text{Aut}(\mathcal{A}_{T_0}) \leq \text{Hol}(T_0).$$

Moreover, by the above $\Gamma^{T_i} \leq \text{Hol}(T_i)$ for all $i > 0$, because the sections T_1, \dots, T_k are primitive. Thus

$$\Gamma^S \leq \prod_{i=0}^k \Gamma^{T_i} \leq \prod_{i=0}^k \text{Hol}(T_i) = \text{Hol}(S).$$

But then Γ^S is obviously a unique subgroup of $\text{Hol}(S)$ in the set $\mathcal{M}(\mathcal{A}_S)$. Thus $\Gamma^S = \text{Hol}_{\mathcal{A}}(S)$. ■

4 Excluding families 1, 2 and 4

In the end of this section we prove the following theorem showing that any minimal non-schurian quasidense S-ring \mathcal{A} over a cyclic group G contains two distinct minimal \mathcal{A} -groups and two distinct maximal \mathcal{A} -groups, the relationship between which, is as in Fig. 1.

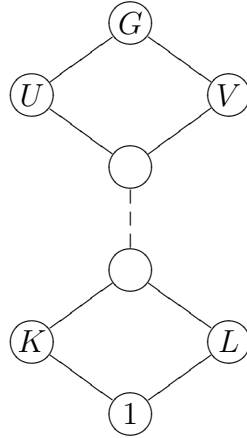


Figure 1:

Theorem 4.1 *Let \mathcal{A} be a minimal non-schurian quasidense S-ring over a cyclic group G of order n belonging to one of five families (1). Then*

- (1) n belongs to the third or fifth families,
- (2) there exist distinct \mathcal{A} -groups K, L of prime orders and distinct \mathcal{A} -groups V, U of prime indices such that $LK \leq U \cap V$ and \mathcal{A} is a proper U/L -wreath product.

We begin with studying minimal non-schurian circulant S-rings. In the following statement we establish general properties of them.

Lemma 4.2 *Let \mathcal{A} be a minimal non-schurian S-ring over a cyclic group G . Then*

- (1) \mathcal{A} is a proper generalized wreath product,
- (2) if \mathcal{A} is a proper U/L -wreath product, then $\text{rad}(\mathcal{A}_{U/L}) \neq 1$; moreover, $|\text{rad}(\mathcal{A}_{U/L})| > 2$ whenever $\mathcal{A}_{U/L}$ is cyclotomic.

Proof. By Corollary A4.3 we can assume that $\text{rad}(\mathcal{A}) \neq 1$. So statement (1) follows from Theorem A4.1. The first part of statement (2) follows from the minimality of \mathcal{A} , Theorem A1.3 and Corollary A1.4. Similarly, to prove the second part of this statement it suffices to verify that the S-ring $\mathcal{A}_{U/L}$ is the tensor product of a normal S-ring and S-rings of rank 2 whenever it is cyclotomic and its radical is of order 2. However, under this condition the criterion of normality [2, Theorem 6.1] implies that $\mathcal{A}_{U/L}$ is not normal if and only if it is the tensor product one factor of which is an S-ring of rank 2 (over a cyclic group of prime order). Thus the required statement follows by induction. ■

By statement (1) of Lemma 4.2 any minimal non-schurian circulant S-ring is a proper generalized wreath product. The following important theorem shows that in the quasidense case both operands are also proper generalized wreath products.

Theorem 4.3 *Let \mathcal{A} be a minimal non-schurian quasidense S-ring over a cyclic group G of order n belonging to one of families (1). Then $\text{rad}(\mathcal{A}_U) \neq 1$ and $\text{rad}(\mathcal{A}_{G/L}) \neq 1$ whenever \mathcal{A} is a proper U/L -wreath product.*

Proof. Let \mathcal{A} be a proper U/L -wreath product. Suppose on the contrary that $\text{rad}(\mathcal{A}_T) = 1$ where $T \in \{U, G/L\}$. Then the quasidensity of \mathcal{A} implies by Theorem 3.1 that the S-ring \mathcal{A}_T , and hence the S-ring \mathcal{A}_S with $S = U/L$,

is cyclotomic. By the minimality of \mathcal{A} and Lemma 4.2 this implies that $|\text{rad}(A_S)| > 2$. Thus n does not belong to the fourth and the fifth families, because otherwise $|S|$ is either prime, or 4, or the product of two distinct primes. Moreover, from Theorem 7.3 for $G = T$ it follows that $S_l = 1$ for some odd prime divisor l of $|T|$. Thus n does not belong to the first family. In the remaining two cases the prime l coincides with p , because otherwise $l = q$, and hence $|S|$ divides $2p$ which is impossible by above. This proves the following lemma.

Lemma 4.4 *Under the above assumptions we have $n = pq^k$ or $n = 2pq^k$, and $p \neq 2$. Moreover,*

- (1) *if $\text{rad}(\mathcal{A}_U) = 1$, then $L_p \neq 1$,*
- (2) *if $\text{rad}(\mathcal{A}_{G/L}) = 1$, then $(G/U)_p \neq 1$.■*

Let $\text{rad}(\mathcal{A}_U) = 1$. Assume that either $q = 2$, or $G_{2'}$ is not an \mathcal{A} -group. By Lemma 4.4 the number $|G/L|$ is a power of q or twice a power of q . So by the assumption there is a unique maximal $\mathcal{A}_{G/L}$ -group, say U'/L . Therefore $U' \geq U$, and hence \mathcal{A} is the U'/L -wreath product. Denote by L' a maximal possible \mathcal{A} -group containing L for which \mathcal{A} is the U'/L' -wreath product. Then the uniqueness of U' implies that $\text{rad}(\mathcal{A}_{G/L'}) = 1$. (Indeed, otherwise by Theorem A4.1 the S-ring \mathcal{A} is the U'/L'' -wreath product for some $L'' > L'$, which contradicts the maximality of L'). Then by statement (2) of Lemma 4.4 we conclude that $(G/U')_p \neq 1$. Taking into account that $L'_p \geq L_p \neq 1$, we conclude that p^2 divides n which is impossible by Lemma 4.4. This proves the first part of the following lemma (the second one is proved in a similar way).

Lemma 4.5 *We have $n = 2pq^k$ and $q \neq 2$. Moreover,*

- (1) *if $\text{rad}(\mathcal{A}_U) = 1$, then $G_{2'}$ is an \mathcal{A} -group,*
- (2) *if $\text{rad}(\mathcal{A}_{G/L}) = 1$, then G_2 is an \mathcal{A} -group.■*

To complete the proof of Theorem 4.3 we come to a contradiction under the assumption $T = U$ (the remaining case $T = G/L$ can be proved in a similar way). In this case we observe that $U' := G_{2'}$ is an \mathcal{A} -group by Lemma 4.5. We claim that

$$\mathcal{A} = \mathcal{A}_{U'} \wr_{U'/L} \mathcal{A}_{G/L}. \quad (6)$$

Indeed, since \mathcal{A} is the U/L -wreath product, it suffices to verify that $U' \geq U$. Suppose on the contrary that this is not true. Then the number $|U|$ must be even. This implies that G_2 is an \mathcal{A} -group (we used the fact that the S-ring \mathcal{A}_U is cyclotomic, and hence dense). Since $p \neq 2$ and $q \neq 2$, this shows that G_2 is the \mathcal{A} -complement of U' . Therefore by Corollary 7.2 we conclude that $\mathcal{A} = \mathcal{A}_{U'} \otimes \mathcal{A}_{G_2}$. By the minimality of \mathcal{A} this implies that the S-ring \mathcal{A} is schurian. The obtained contradiction proves equality (6).

After increasing the group L in (6) (if necessary), we can assume that it is a maximal possible \mathcal{A} -group with that property. Then

$$(G/L)_p = 1 \quad \text{and} \quad (G/L)_2 \notin \mathcal{G}(\mathcal{A}_{G/L}). \quad (7)$$

The first equality follows from Lemma 4.4. To prove the second one suppose on the contrary that the group $(G/L)_2$ is the $\mathcal{A}_{G/L}$ -complement of U'/L . Therefore by Corollary 7.2 we conclude that $\mathcal{A}_{G/L} = \mathcal{A}_{U'/L} \otimes \mathcal{A}_{(G/L)_2}$. By the minimality of \mathcal{A} and Theorem 7.5 this implies that the S-ring \mathcal{A} is schurian. The obtained contradiction proves (7).

Due to the quasidensity of \mathcal{A} formula (7) implies that there is the only minimal $\mathcal{A}_{G/L}$ -group, say L'/L , and $|L'/L| = q$. We claim that

$$\mathcal{A}_{G/L} = \mathcal{A}_{U'/L} \wr_{U'/L'} \mathcal{A}_{G/L'}. \quad (8)$$

Indeed, otherwise by Corollary 3.3 there exists an $\mathcal{A}_{G/L}$ -group H/L such that $H/L \not\leq U'/L$ and $(H/L)_{q'}$ is an $\mathcal{A}_{G/L}$ -group. However, it is easily seen that in our case $(H/L)_{q'} = (G/L)_2$, which contradicts the second relation in (7). The obtained contradiction proves (8).

Equalities (6) and (8) show that the S-ring \mathcal{A} is the U'/L' -wreath product. However, this is impossible by the maximality of L . ■

Proof of Theorem 4.1. Statement (1) immediately follows from statement (2). To prove the latter we observe that by Lemma 4.2 the S-ring \mathcal{A} is a proper U/L -wreath product for some \mathcal{A} -groups U and L . By the quasidensity of \mathcal{A} we can assume that L is of prime order and U is of prime index. Denote by \tilde{U} a minimal subgroup of U such that the S-ring \mathcal{A} is the \tilde{U}/L -wreath product. Then by Theorem 4.3 the S-ring $\mathcal{A}_{\tilde{U}}$ is a proper U'/K -wreath product for some \mathcal{A} -groups U' and K . Again we can assume that K is of prime order. By the minimality of \tilde{U} we conclude that $K \neq L$. Besides, $KL \leq U$ because $L \leq U$ and $K \leq U' < \tilde{U} \leq U$.

Similarly, denote by \tilde{L} a maximal subgroup of U that contains L and such that the S-ring \mathcal{A} is the U/\tilde{L} -wreath product. Then again by Theorem 4.3 the S-ring $\mathcal{A}_{G/\tilde{L}}$ is a proper V/L' -wreath product for some \mathcal{A} -groups V and L' such that V is of prime index in G . By the maximality of \tilde{L} we conclude that $V \neq U$. Besides, obviously $V \geq L$. To complete the proof it suffices to note that $K \leq V$. Indeed, if this is not true, then any nontrivial basic set of $\mathcal{A}_{G/\tilde{L}}$ inside $K\tilde{L}/\tilde{L}$, is outside V/\tilde{L} and has trivial radical (because $|K\tilde{L}/\tilde{L}| = |K|$ is prime), which is impossible. ■

5 Excluding families 3 and 5 for $p \neq 2$

In the end of this section we prove the following theorem that will enable us to exclude the cases in the title.

Theorem 5.1 *Let \mathcal{A} be a minimal non-schurian quasidense S-ring over a cyclic group G of order n belonging to the third or fifth of families (1). Then*

- (1) $p = 2$,
- (2) \mathcal{A} is both U/L - and V/K -wreath product where K, L, U, V are \mathcal{A} -groups defined by
 - (1) $|K| = 2, |L| = q, |U| = 2qr, |V| = 4q$ for $n = 4qr$,
 - (2) $|K| = 2, |L| = q, |U| = 2q^k, |V| = 4q^{k-1}$ for $n = 4q^k$,

with q and r distinct odd primes and $k \geq 2$.

Throughout the rest of the section \mathcal{A} denotes an S-ring satisfying the hypothesis of Theorem 5.1. It is also assumed that we are given \mathcal{A} -groups K, L, U, V for which statement (2) of Theorem 4.1 holds.

Theorem 5.2 *The number $|U/L|$ is even.*

Proof. Suppose on the contrary that $|U/L|$ is odd. Then either $|L| = 2$, or $|G/U| = 2$. Let us consider the former case, the latter one can be proved similarly. We are to find an \mathcal{A} -group U' such that $L \leq U' \leq U$ and

$$\mathcal{A} = \mathcal{A}_{U'} \wr_{U'/L} \mathcal{A}_{G/L} \quad \text{and} \quad (U')_{2'} \in \mathcal{G}(\mathcal{A}). \quad (9)$$

Indeed, if such a group does exist, then by Corollary 7.2 with $G = U'$, $H = (U')_{2'}$, $S = L/1$ and $T = U'/L$, we obtain that $\mathcal{A}_U = \mathcal{A}_L \otimes \mathcal{A}_{U'/L}$. By the minimality of \mathcal{A} and Theorem 7.5 with $U = U'$ this implies that the S-ring \mathcal{A} is schurian which is not true.

In the case $n = 2pqr$ set $U' = U$. Then the left-hand side relation in (9) is obvious. To prove the other one we observe that by Theorem A11.4 the number $|U|$ is the product of three primes, the S-ring \mathcal{A}_U is not a proper wreath product and $\mathcal{A}_{U/L}$ is a proper wreath product. So, since the number $|U/L|$ is odd, the hypothesis of Lemma A11.2 is satisfied for $\mathcal{A} = \mathcal{A}_U$, $S = U/L$ and $r = 2$. By this lemma we obtain that $U_{2'} \in \mathcal{G}(\mathcal{A}_U)$, and we are done.

In the case $n = 2pq^k$ set U' to be a minimal \mathcal{A} -subgroup of U , for which the first relation in (9) holds. We can assume that

$$(U')_p \neq 1. \quad (10)$$

Indeed, otherwise $|U'| = 2q^i$ for some i . Moreover, the minimality of U' implies that the S-ring $\mathcal{A}_{U'}$ is not the U''/L -wreath product where U'' is the subgroup of U' of index q . Then by Corollary 3.3 with $p = 2$ there exists an $\mathcal{A}_{U'}$ -group H_1 such that $(U')_q \leq H_1 \leq (U')_{2'}$. It follows that $H_1 = (U')_q$, and the second relation in (9) holds.

From (10) it follows that $|G/V| = p$, and taking into account that $|U/L|$ is odd, also that p is odd. Therefore $U' \cap V \neq U'$. By the minimality of U' this implies that the S-ring $\mathcal{A}_{U'}$ is not a $(U' \cap V)/L$ -wreath product. So by Corollary 3.3 with $(p, q) = (2, p)$ there exists an \mathcal{A} -group H_1 such that

$$(U')_p \leq H_1 \leq (U')_{2'}. \quad (11)$$

Denote by H a maximal \mathcal{A} -subgroup of U' that contains H_1 . Then due to (11) we have $|U'/H| \in \{2, q\}$. If $|U'/H| = 2$, then the second relation in (9) holds and we are done. Finally, if $|U'/H| = q$ then due to the minimality of U' the S-ring $\mathcal{A}_{U'}$ is not an H/L -wreath product. By Corollary 3.3 with $(p, q) = (2, q)$ there exists an \mathcal{A} -group H such that

$$(U')_q \leq H_2 \leq (U')_{2'}. \quad (12)$$

Thus by (11) and (12) we have $(U')_{2'} = H_1 H_2$, and hence $(U')_{2'} \in \mathcal{G}(\mathcal{A})$. ■

Theorem 5.3 *The order of G is divisible by 4.*

Proof. Suppose first that $n = 2pqr$. Then by statement (1) of Theorem A11.4 (where the lattice of \mathcal{A} -groups is found) there are exactly two maximal \mathcal{A} -groups and exactly two minimal \mathcal{A} -groups; the former are of prime index whereas the latter are of prime order. From statement (2) of Theorem 4.1 it follows that these groups are U , V and K , L respectively, and also that

$$|U/L| \cdot |V/K| = |G|. \quad (13)$$

On the other hand, we claim that the S-rings \mathcal{A}_U and $\mathcal{A}_{G/L}$ are non-normal. Indeed, otherwise by statement (3) of Theorem A11.4 the number $|U/L|$ is a square of a prime. In our case this is possible only for $p = 2$. But in this case $|U/L| = 4$ which is impossible by the latter theorem. The claim is proved. So by statement (5) of the same theorem the S-ring \mathcal{A} is the V/K -wreath product. Then by Theorem 5.2 of this paper the number $|V/K|$ is even. Thus 4 divides $|G|$.

Let $n = 2pq^k$. Suppose on the contrary that $p \neq 2$. Then since q is odd and $|U/L|$ is even (Theorem 5.2), from statement (2) of Theorem 4.1 it follows that $|G/V| = 2$ or $|K| = 2$. Let us consider the former case, the latter one can be proved similarly. In this case $|V|$ is odd. Therefore by Theorem 5.2 the S-ring \mathcal{A} is neither V/K - nor V/L -wreath product. So by Corollary 3.3 with $(p, q) = (2, p)$ and $(p, q) = (2, q)$ there exist \mathcal{A} -groups H_1 and H_2 such that

$$G_2 \leq H_1 \leq G_{p'} \quad \text{and} \quad G_2 \leq H_2 \leq G_{q'}.$$

Thus $G_2 = H_1 \cap H_2$, and hence G_2 is an \mathcal{A} -group. By Corollary 7.2 this implies that $\mathcal{A} = \mathcal{A}_{G_2} \otimes \mathcal{A}_V$ which is impossible by the minimality of \mathcal{A} . ■

Theorem 5.4 *Without loss of generality we can assume that 4 does not divide $|U/L|$.*

Proof. By Theorem 5.3 we have $p = 2$. So $n = 4qr$ or $n = 4q^k$. In the former case $|U/L|$ is divisible by 4 only if $|U/L| = 4$. However, in this case the S-ring $\mathcal{A}_{U/L}$ is cyclotomic and $|\text{rad}(\mathcal{A}_{U/L})| \leq 2$, which contradicts statement (2) of Lemma 4.2. Thus we can assume that $n = 4q^k$ and 4 divides $|U/L|$. Then from statement (2) of Theorem 4.1 it follows that

$$|G/U| = |L| = q \quad \text{and} \quad |G/V| = |K| = 2.$$

Therefore it suffices to verify that the S-ring \mathcal{A} is either U/K - or V/L -wreath product. Suppose on the contrary that this is not true. Then by Corollary 3.3 with $(p, q) = (2, q)$ and $(p, q) = (q, 2)$ there exist \mathcal{A} -groups H_1 and H_2 such that

$$G_q \leq H_1 \leq G_{2'} \quad \text{and} \quad G_2 \leq H_2 \leq G_{q'}.$$

It follows that $H_1 = G_q$ and $H_2 = G_2$. Thus G_q and G_2 are \mathcal{A} -groups. By the quasidensity of \mathcal{A} this implies that \mathcal{A} is dense.

Denote by \tilde{U} the minimal \mathcal{A} -subgroup of U , for which the S-ring \mathcal{A} is the \tilde{U}/L -wreath product. Then by Theorem 4.3 with $U = \tilde{U}$ the radical of the ring $\mathcal{A}_{\tilde{U}}$ is nontrivial. Since this S-ring is dense, from [6, Theorem 3.4] (see also statement (1) of [2, Theorem 5.4]) it follows that it is a U'/L' -wreath product where the number $|L'| = |\tilde{U}/U'|$ is the greatest prime divisor of $|\text{rad}(\mathcal{A}_{\tilde{U}})|$. By the minimality of the group \tilde{U} we conclude that this prime divisor is equal to 2. Thus

$$L' = K = \text{rad}(\mathcal{A}_{\tilde{U}}) \quad \text{and} \quad |\tilde{U}/U'| = 2. \quad (14)$$

By Corollary 7.2 we have $\mathcal{A}_{U'} = \mathcal{A}_K \otimes \mathcal{A}_{U'_q}$ and $\mathcal{A}_{\tilde{U}/K} = \mathcal{A}_{\tilde{U}_2/K} \otimes \mathcal{A}_{U'/K}$. Since $\mathcal{A}_{\tilde{U}}$ is the U'/K -wreath product this implies that

$$\mathcal{A}_{\tilde{U}} = \mathcal{A}_{\tilde{U}_2} \otimes \mathcal{A}_{\tilde{U}_q}. \quad (15)$$

Therefore $\mathcal{A}_{\tilde{U}/L} \cong \mathcal{A}_{\tilde{U}_2} \otimes \mathcal{A}_{\tilde{U}_q/L}$. Moreover, the S-ring $\mathcal{A}_{\tilde{U}_2}$ being a dense S-ring over a cyclic group of order 4, is cyclotomic and $|\text{rad}(\mathcal{A}_{\tilde{U}_2})| \leq 2$. Finally, by Corollary 7.4 the S-ring $\mathcal{A}_{\tilde{U}_q/L}$ has trivial radical because by (14) and (15) so is the S-ring $\mathcal{A}_{\tilde{U}_q}$. Therefore the latter S-ring is cyclotomic by Theorem 3.1. Thus the S-ring $\mathcal{A}_{\tilde{U}/L}$ is cyclotomic and $|\text{rad}(\mathcal{A}_{\tilde{U}/L})| \leq 2$ which contradicts statement (2) of Lemma 4.2. ■

Proof of Theorem 5.1. By Theorem 4.1 we can assume that the hypothesis under which Theorems 5.2, 5.3 and 5.4 were proved, holds for the S-ring \mathcal{A} . Then statement (1) immediately follows from Theorem 5.3. Thus

$$n = 4qr \quad \text{or} \quad n = 4q^k$$

where q and r distinct odd primes and $k \geq 2$. To prove statement (2) choose the groups K, L, U, V as above. Then obviously $|KL| = 2q$ when $n = 4q^k$. The same is also true for $n = 4qr$ after interchanging q and r (if necessary). By Theorems 5.2 and 5.4 we can also assume $|U/L| = 2 \pmod{4}$.

Let $n = 4qr$. Then by the above assumptions the number $|U/L|$ is not a prime square. By statement (3) of Theorem A11.4 this implies that neither of the S-rings \mathcal{A}_U and $\mathcal{A}_{G/L}$ is normal. Therefore by statement (5) of that theorem the S-ring \mathcal{A} is the V/K -wreath product. Besides $|V/K| = 2 \pmod{4}$. Thus without loss of generality we can assume that $|K| = 2$. Then $|L| = q$ and $|G/U| = 2$. It follows that $|U| = 2qr$ and $|V| = 4q$, which completes the proof in this case.

Let $n = 4q^k$. Then by the above assumptions one of the following holds:

- (1) $|K| = 2, |L| = q, |U| = 2q^k, |V| = 4q^{k-1}$,
- (2) $|K| = q, |L| = 2, |U| = 4q^{k-1}, |V| = 2q^k$.

Thus it suffices to verify that \mathcal{A} is the V/K -wreath product. Suppose that this is not true.

In case (1) by Corollary 3.3 with $p = 2$ there exists an \mathcal{A} -group H_1 such that $G_q \leq H_1 \leq G_{2'}$. It follows that $H_1 = G_q$, and hence G_q is an \mathcal{A} -group. By Corollary 7.2 this implies that

$$\mathcal{A}_U = \mathcal{A}_{G_q} \otimes \mathcal{A}_K.$$

Denote by H the maximal \mathcal{A} -group such that $L \leq H \leq G_q$ and $\text{rad}(\mathcal{A}_H) = 1$. Set $U' = HK$. Then the radical of any basic set inside $U \setminus U'$ contains L . Since the same is true also for any basic set outside U , the S-ring \mathcal{A} is the U'/L -wreath product. Besides, $\text{rad}(\mathcal{A}_{U'}) = 1$ because $\mathcal{A}_{U'} = \mathcal{A}_H \otimes \mathcal{A}_K$ and $|K| = 2$. By statement (2) of Theorem 7.3 with $G = U'$ this implies that $\text{rad}(\mathcal{A}_{U'/L}) = 1$. However, this contradicts statement (2) of Lemma 4.2. Thus case (1) is impossible.

In case (2) one can similarly prove that G_2 is an \mathcal{A} -group, and $\mathcal{A}_{G/L} = \mathcal{A}_{G_2/L} \otimes \mathcal{A}_{V/L}$. It is also easily seen that any basic set outside U is highest in \mathcal{A} or in \mathcal{A}_V . Therefore the S-ring \mathcal{A} is the U/L' -wreath product where $L' = \text{rad}(\mathcal{A}_V)$. Besides, $\text{rad}(\mathcal{A}_{G/L'}) = 1$ because $\mathcal{A}_{G/L'} = \mathcal{A}_{H/L'} \otimes \mathcal{A}_{V/L'}$ and $|H/L'| = 2$, where $H = G_2L'$. By statement (2) of Theorem 7.3 with $G = G/L'$ this implies that $\text{rad}(\mathcal{A}_{U/L'}) = 1$. However, this is impossible by statement (2) of Lemma 4.2. ■

6 Proof of Theorem 1.4

Let \mathcal{A} be a minimal non-schurian S-ring over a cyclic group G of order n . Suppose on the contrary that n belongs to one of families (1). Since any divisor of n also belongs to one of these families, by statement (3) of Lemma 4.2 without loss of generality we can assume that \mathcal{A} is quasidense. Then by Theorems 4.1 and 5.1 we have $n = 4qr$ or $n = 4q^k$, and \mathcal{A} is both U/L - and V/K -wreath product where K, L, U, V are \mathcal{A} -groups defined by

- (1) $|K| = 2, |L| = q, |U| = 2qr, |V| = 4q$ for $n = 4qr$,
- (2) $|K| = 2, |L| = q, |U| = 2q^k, |V| = 4q^{k-1}$ for $n = 4q^k$,

with q and r distinct odd primes and $k \geq 2$. In both cases we will verify that the hypothesis of Theorem 8.1 is satisfied for some \mathcal{A} -groups so that the generalized wreath product for \mathcal{A} defined there is proper. Then by that theorem \mathcal{A} is schurian because due to the minimality of \mathcal{A} so are the operands of this product. Contradiction.

Suppose that we are in case (1). Set $H_1 = H_2 = H$ where $H = KL = U \cap V$. First, we observe that relations (23) and (24) are obviously satisfied. Furthermore, $|U/L| = 2r$ is not a prime square, and hence by statement (3) of Theorem A11.4 the S-rings \mathcal{A}_U and $\mathcal{A}_{G/L}$ are not normal. By statement (4) of that theorem this implies that these S-rings are respectively the H/K - and V/H -wreath products. Besides, condition (1) of Theorem 8.1 is trivially satisfied because the underlying groups of the S-rings $\mathcal{A}_{H/K}, \mathcal{A}_{U/H}, \mathcal{A}_{V/H}$ and $\mathcal{A}_{H/L}$, are of prime orders, whereas condition (2) is satisfied because $|K| = |G/U| = 2$. Thus the hypothesis of Theorem 8.1 is satisfied.

Suppose that we are in case (2). To define the \mathcal{A} -groups from the hypothesis of Theorem 8.1 we have to do preliminary work. Set M to be the minimal \mathcal{A} -subgroup of G that contains G_2 , and N to be the maximal \mathcal{A} -subgroup of G_q . We claim that

$$G_2 \neq M, \quad M_q \leq N, \quad N \neq G_q. \quad (16)$$

Indeed, if $G_2 = M$, then the radical of the highest basic set in G_2 has trivial q -part. However, this is impossible because \mathcal{A} is the U/L -wreath product. Similarly, if $N = G_q$, then the radical of the highest basic set in G_q has trivial 2-part. However, this is impossible because \mathcal{A} is the V/K -wreath product.

To prove the rest we observe that by Theorem 5.2 the S-ring \mathcal{A} is not a U/K -wreath product. Then by statement (1) of Corollary 3.3 with $p = 2$ there exists an \mathcal{A} -group H such that $G_2 \leq H$ and H_q is an \mathcal{A} -group. So $M_q \leq H_q$ and $N \geq H_q$ by the choice of M and N respectively. This proves the claim.

Let us verify that the hypothesis of Theorem 8.1 is satisfied for \mathcal{A} -groups $K, \tilde{L}, M, N, U, \tilde{V}$ where

$$\tilde{L} = M \cap N \quad \text{and} \quad \tilde{V} = MN.$$

Then obviously $\tilde{L} \leq N$ and $M \leq \tilde{V}$. Moreover, from (16) it also follows that $H_1 \leq H_2$ where $H_1 = K\tilde{L}$ and $H_2 = \tilde{V} \cap U$. Since also

$$H_2 = K \times N \quad \text{and} \quad G/H_1 = M/H_1 \times U/H_1, \quad (17)$$

the relations (23) and (24) hold. A part of the \mathcal{A} -group lattice is given at Fig. 2.

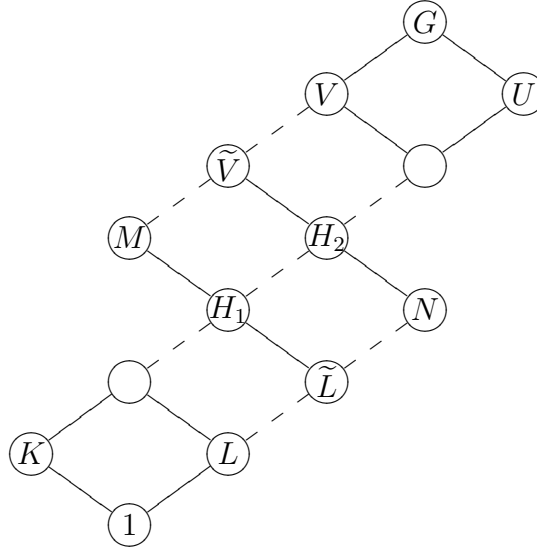


Figure 2:

To verify the rest of the hypothesis of Theorem 8.1 we observe that condition (2) is satisfied because $|K| = |G/U| = 2$. We claim that

$$\mathcal{A} = \mathcal{A}_U \wr_{U/\tilde{L}} \mathcal{A}_{G/\tilde{L}}. \quad (18)$$

Suppose on the contrary that this is not true. Then there exists a basic set X outside U such that $\text{rad}(X)_q < \tilde{L}$. Then obviously $M' := G_2 \text{rad}(X)$ is a proper subgroup of M that contains G_2 , which is an \mathcal{A} -group by Corollary 3.2. However, this contradicts the minimality of M . Next, let us verify that

$$\mathcal{A}_U = \mathcal{A}_{H_2/K} \wr_{H_2/K} \mathcal{A}_{U/K} \quad \text{and} \quad \mathcal{A}_{G/\tilde{L}} = \mathcal{A}_{\tilde{V}/\tilde{L}} \wr_{\tilde{V}/H_1} \mathcal{A}_{G/H_1}. \quad (19)$$

To prove the first equality suppose on the contrary that the S-ring \mathcal{A}_U is not the H_2/K -wreath product. Then by statement (1) of Corollary 3.3 with $p = 2$ there exists an \mathcal{A}_U -group $H \not\leq H_2$ such that H_q is an \mathcal{A} -group. However, this is impossible by the maximality of N . The second equality can be proved in a similar way. Thus by Remark 8.2 we only have to prove that

$$\text{rad}(\mathcal{A}_{H_2/K}) = 1, \quad \text{rad}(\mathcal{A}_{U/H_1}) = 1, \quad \text{rad}(\mathcal{A}_{H_2/H_1}) = 1. \quad (20)$$

We observe that the third equality follows from the first one and Corollary 7.4 for $G = H_2/K$. To prove the first equality in (20) suppose on the contrary that $\text{rad}(\mathcal{A}_{H_2/K}) > 1$. To get a contradiction we use the idea from the proof of case (1) in Theorem 5.1. First, we observe that by Corollary 7.2 we have

$$\mathcal{A}_{H_2} = \mathcal{A}_N \otimes \mathcal{A}_K.$$

Set $U' = N'K$ where N' is the maximal \mathcal{A} -group such that $L \leq N' \leq N$ and $\text{rad}(\mathcal{A}_{N'}) = 1$. Then $N' < N$ by the above supposition and the fact that $\mathcal{A}_N \cong \mathcal{A}_{H_2/K}$. Next, let X be a basic set outside U' . Then $L \leq \text{rad}(X)$ for $X \subset G \setminus U$ because \mathcal{A} is the U/L -wreath product and for $X \subset H_2 \setminus U'$ by the definition of U' . The same is also true for $X \subset U \setminus H_2$. Indeed, otherwise set $Q = \langle X \rangle / \text{rad}(X)$ and S to be the image of the section H_2/K in Q . Then $\text{rad}(\mathcal{A}_S) = 1$ by Theorem 7.3 applied to the S-ring \mathcal{A}_Q and the section S . On the other hand, $\text{rad}(\mathcal{A}_S) > 1$ because $\text{rad}(\mathcal{A}_{H_2/K}) > 1$ and $\text{rad}(X) \leq K$. Contradiction. Thus the S-ring \mathcal{A} is the U'/L -wreath product. Besides, $\text{rad}(\mathcal{A}_{U'}) = 1$ because $\text{rad}(\mathcal{A}_{N'}) = 1$ and $\mathcal{A}_{U'} = \mathcal{A}_{N'} \otimes \mathcal{A}_K$. By statement (2) of Theorem 7.3 for $G = U'$ this implies that $\text{rad}(\mathcal{A}_{U'/L}) = 1$. However, this contradicts statement (2) of Lemma 4.2. The second equality in (20) is proved similarly following the proof of case (2) in Theorem 5.1. ■

7 Auxiliary statements on S-rings

Given an S-ring \mathcal{A} over a group G we define an \mathcal{A} -complement of an \mathcal{A} -group H to be an \mathcal{A} -group H' such that $G = H \times H'$. When the group G is

cyclic, the group H' is obviously uniquely determined.

Theorem 7.1 *Let \mathcal{A} be an S -ring over a cyclic group G . Suppose that an \mathcal{A} -group H has an \mathcal{A} -complement and $\mathcal{A}_S = \mathbb{Z}S$ where S is an \mathcal{A} -section projectively equivalent to G/H . Then given an \mathcal{A} -section T projectively equivalent to H the S -rings \mathcal{A} and $\mathcal{A}_S \otimes \mathcal{A}_T$ are Cayley isomorphic.*

Proof. Denote by H' the \mathcal{A} -complement of H . Then obviously $H'/1$ and G/H are respectively the smallest and greatest \mathcal{A} -sections in the class of projectively equivalent \mathcal{A} -sections that contains G/H . This implies that the section S is projectively equivalent to (in fact, a multiple of) $H'/1$. By Theorem A3.2 the S -rings \mathcal{A}_S and $\mathcal{A}_{H'}$ as well as \mathcal{A}_T and \mathcal{A}_H are Cayley isomorphic. Thus without loss of generality we can assume that $S = H'/1$ and $T = H/1$. Then $\mathcal{A}_{H'} = \mathbb{Z}H'$, and hence

$$\text{rk}(\mathcal{A}) = |H'| \text{rk}(\mathcal{A}_H) = \text{rk}(\mathcal{A}_{H'}) \text{rk}(\mathcal{A}_H).$$

Since also $\mathcal{A} \geq \mathcal{A}_H \otimes \mathcal{A}_{H'}$ by Lemma A2.1, we have $\mathcal{A} = \mathcal{A}_H \otimes \mathcal{A}_{H'}$. ■

Corollary 7.2 *Theorem 7.1 remains true with the condition $\mathcal{A}_S = \mathbb{Z}S$ replaced by $|S| = 2$. ■*

Some parts of the following statement appeared in a number of papers. Here we formulate it in a more or less general form because it is used throughout the paper several times.

Theorem 7.3 *Let \mathcal{A} be a cyclotomic S -ring with trivial radical over a cyclic group G . Suppose that S is an \mathcal{A} -section such that $S_p \neq 1$ for any odd prime divisor p of $|G|$. Then*

- (1) $|\text{rad}(\mathcal{A}_S)| \leq 2$,
- (2) $|\text{rad}(\mathcal{A}_S)| = 1$ unless $|S_2| = 4$.

Proof. By [6, Lemma 3.5] given a set $X \in \mathcal{S}(\mathcal{A})$ with $\text{rad}(X) = 1$ and a prime p such that p^2 divides $m = |\langle X \rangle|$, we have $\text{rad}(X^p) = 1$ unless $p = 2$ and $m = 8m'$ with m' odd. This shows that $\text{rad}(\mathcal{A}_U) = 1$ where U is the subgroup of G of index p , unless $p = 2$ and $|G| = 8m'$ with m' odd. Since $\mathcal{A}_{G/L} \cong \mathcal{A}_U$ where L is the subgroup of G of order p , we have $\mathcal{A}_{G/L} = 1$ under

the same conditions. Thus recursively applying these results we reduces the lemma to the case

$$|S_2| \leq 4, \quad |G_2| \leq 8, \quad S_{2'} = G_{2'}.$$

However, from [6, Proposition 3.1] with $m = |G|$ and $l = |G_{2'}|$ it follows that $\text{rad}(\mathcal{A}_{G_{2'}}) = 1$. On the other hand, $\text{rad}(\mathcal{A}_S) \leq \text{rad}(\mathcal{A}_{S_2}) \text{rad}(\mathcal{A}_{S_{2'}})$ because $\mathcal{A}_S \geq \mathcal{A}_{S_2} \otimes \mathcal{A}_{S_{2'}}$ (see Lemma A2.2). Thus $\text{rad}(\mathcal{A}_S) \leq \text{rad}(\mathcal{A}_{S_2})$, and we are done. ■

From Theorems A4.1, A4.2 and 7.3 we immediately obtain the following useful result.

Corollary 7.4 *Let \mathcal{A} be an S-ring with trivial radical over a cyclic p -group, p odd. Then $\text{rad}(\mathcal{A}_S) = 1$ for any \mathcal{A} -section S . ■*

The following statement gives a necessary and sufficient condition for the schurity of an U/L -wreath product when the section U/L is one of two sections forming an isolated pair of sections in the corresponding S-ring (see Definition A6.1).

Theorem 7.5 *Let $\mathcal{A} = \mathcal{A}_U \wr_{U/L} \mathcal{A}_{G/L}$ be an S-ring over a cyclic group G . Suppose that either $\mathcal{A}_U \cong \mathcal{A}_L \otimes \mathcal{A}_{U/L}$ or $\mathcal{A}_{G/L} \cong \mathcal{A}_{U/L} \otimes \mathcal{A}_{G/U}$. Then the S-ring \mathcal{A} is schurian if and only if so are the S-rings \mathcal{A}_U and $\mathcal{A}_{G/L}$.*

Proof. The necessity is obvious because given an \mathcal{A} -section S the S-ring \mathcal{A}_S is schurian whenever so is \mathcal{A} . Let us prove the sufficiency under the assumption $\mathcal{A}_U \cong \mathcal{A}_L \otimes \mathcal{A}_{U/L}$ (the rest can be proved analogously). Denote by $f : U \rightarrow L \times (U/L)$ the corresponding Cayley isomorphism. Then $L^f = L$ and $H^f = U/L$ for a uniquely determined \mathcal{A} -group H . It follows that $\mathcal{A}_U = \mathcal{A}_L \otimes \mathcal{A}_H$. Set

$$\Delta_0 = \text{Aut}(\mathcal{A}_{G/L}) \quad \text{and} \quad \Delta_1 = \text{Aut}(\mathcal{A}_L) \otimes \Delta_H.$$

where Δ_H is the full $f^{U/L}$ -preimage of the group $(\Delta_0)^{U/L}$ in the group $\text{Aut}(\mathcal{A}_H)$. Clearly,

$$(G/L)_{\text{right}} \leq \Delta_0, \quad U_{\text{right}} \leq \Delta_1, \quad (\Delta_0)^{U/L} = (\Delta_1)^{U/L}.$$

Moreover, by the schurity of the S-ring $\mathcal{A}_{G/L}$ the latter group is 2-equivalent to the group $\text{Aut}(\mathcal{A}_{U/L})$. It follows that the groups Δ_H and $\text{Aut}(\mathcal{A}_H)$, and are 2-equivalent. So by the schurity of the S-ring \mathcal{A}_U the groups Δ_1 and $\text{Aut}(\mathcal{A}_U) = \text{Aut}(\mathcal{A}_L) \otimes \text{Aut}(\mathcal{A}_H)$ are also 2-equivalent. Thus by Theorem A1.2 the S-ring \mathcal{A} is schurian and we are done. ■

8 A special generalized wreath product

In this section under special conditions we prove a necessary and sufficient condition for a U/L -wreath product to be schurian when the restriction of it to U/L is also a generalized wreath product. We start with the description of elements of the canonical generalized wreath product introduced in Definition A5.3 ².

Let G be an abelian group and $L \leq U \leq G$. Suppose we are given groups $\Delta_0 \leq \text{Sym}(G/L)$ and $\Delta_1 \leq \text{Sym}(U)$ such that U/L is both Δ_0 - and Δ_1 -section and

$$(G/L)_{\text{right}} \leq \Delta_0, \quad U_{\text{right}} \leq \Delta_1, \quad (\Delta_0)^{U/L} = (\Delta_1)^{U/L}.$$

Then an element of the the canonical generalized wreath product

$$\Gamma = \Delta_1 \wr_{U/L} \Delta_0$$

can explicitly be described as follows. Let us fix bijections $h_X \in (G_{\text{right}})^{U,X}$ where $X \in G/U$. Suppose we are given a permutation $f_0 \in \Delta_0$ and a family $\{f_X \in \Delta_1 : X \in G/U\}$ of permutations such that

$$(f_X)^{U/L} = (h_X)^{U/L} f_0^{X/L} ((h_{X'})^{U/L})^{-1} \quad (21)$$

for all $X \in G/U$ where X' is the U -coset for which $X'/L = (X/L)^{f_0}$. Then obviously there exists a uniquely determined permutation $f \in \text{Sym}(G)$ for which

$$f^{G/L} = f_0 \quad \text{and} \quad f^X = (h_X)^{-1} f_X h_{X'}$$

for all $X \in G/U$. We stress that this permutation depends on the choice of the permutations h_X . Denote it by $\{f_X\} \wr_{U/L} f_0$. Then the definition of the generalized wreath product of permutation groups implies immediately that

$$\Gamma = \{\{f_X\} \wr_{U/L} f_0 : f_0 \in \Delta_0, f_X \in \Delta_1 \text{ for all } X \in G/U\}. \quad (22)$$

Let us turn to the main theorem of this section. Let \mathcal{A} be a quasidense S-ring over a cyclic group G . Suppose we are given \mathcal{A} -groups K, L, M, N, U, V such that $L \leq N, M \leq V$,

$$H_1 := KL \leq U \cap V := H_2 \quad (23)$$

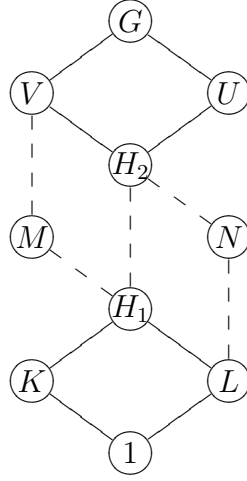


Figure 3:

and also

$$H_2 = K \times N \quad \text{and} \quad G/H_1 = M/H_1 \times U/H_1. \quad (24)$$

The corresponding part of the \mathcal{A} -group lattice is represented in Fig. 3.

Theorem 8.1 *In the above notation suppose that the S-rings \mathcal{A} , \mathcal{A}_U and $\mathcal{A}_{G/L}$ are respectively the U/L -, H_2/K - and V/H_1 -wreath products such that*

- (1) *the S-rings $\mathcal{A}_{H_2/K}$, \mathcal{A}_{U/H_1} and \mathcal{A}_{V/H_1} , $\mathcal{A}_{H_2/L}$ are of trivial radicals,*
- (2) *$\mathcal{A}_K = \mathbb{Z}K$ and $\mathcal{A}_{G/U} = \mathbb{Z}G/U$.*

Then the S-ring \mathcal{A} is schurian if and only if so are the S-rings \mathcal{A}_U and $\mathcal{A}_{G/L}$.

Remark 8.2 *Equalities (24) together with condition (2) imply by Theorem 7.1 that $\mathcal{A}_{V/H_1} \cong \mathcal{A}_{G/U} \otimes \mathcal{A}_{H_2/H_1}$ and $\mathcal{A}_{H_2/L} \cong \mathcal{A}_K \otimes \mathcal{A}_{H_2/H_1}$. Thus in our case the second part of condition (1) is equivalent to the equality $\text{rad}(\mathcal{A}_{H_2/H_1}) = 1$.*

Proof. The necessity is obvious. To prove the sufficiency suppose that the S-rings \mathcal{A}_U and $\mathcal{A}_{G/L}$ are schurian. Set

$$\Gamma_1 = \text{Hol}_{\mathcal{A}}(U/H_1) \quad \text{and} \quad \Gamma_2 = \text{Hol}_{\mathcal{A}}(H_2/L).$$

²The group that was denoted there by Δ_U is denoted here by Δ_1 .

Then obviously $(\Gamma_1)^{H_2/H_1} = (\Gamma_2)^{H_2/H_1}$. So one can define the generalized wreath product $\Delta = \Gamma_2 \wr_{H_2/H_1} \Gamma_1$. Thus by Theorem A1.2 to complete the proof it suffices to find groups $\Delta_1 \in \mathcal{M}(\mathcal{A}_U)$ and $\Delta_0 \in \mathcal{M}(\mathcal{A}_{G/L})$ such that

$$(\Delta_1)^{U/L} = \Delta = (\Delta_0)^{U/L}. \quad (25)$$

To do this we observe that by Theorem 3.5 and due to condition (1) there exist groups $\Gamma_3 \in \mathcal{M}(\mathcal{A}_{U/K})$ and $\Gamma_6 \in \mathcal{M}(\mathcal{A}_{V/L})$ such that

$$(\Gamma_3)^{H_2/K} = \text{Hol}_{\mathcal{A}}(H_2/K), \quad (\Gamma_3)^{U/H_1} = \text{Hol}_{\mathcal{A}}(U/H_1), \quad (26)$$

$$(\Gamma_6)^{V/H_1} = \text{Hol}_{\mathcal{A}}(V/H_1), \quad (\Gamma_6)^{H_2/L} = \text{Hol}_{\mathcal{A}}(H_2/L). \quad (27)$$

Set

$$\Gamma_4 = \text{Hol}_{\mathcal{A}}(H_2), \quad \Gamma_5 = \text{Hol}_{\mathcal{A}}(G/H_1). \quad (28)$$

Then clearly $(\Gamma_4)^{H_2/K} = (\Gamma_3)^{H_2/K}$ and $(\Gamma_6)^{V/H_1} = (\Gamma_5)^{V/H_1}$. Therefore one can define generalized wreath products

$$\Delta_1 = \Gamma_4 \wr_{H_2/K} \Gamma_3 \quad \text{and} \quad \Delta_0 = \Gamma_6 \wr_{V/H_1} \Gamma_5.$$

First, let us prove that $\Delta_1 \in \mathcal{M}(\mathcal{A}_U)$ and $\Delta_0 \in \mathcal{M}(\mathcal{A}_{G/L})$. Indeed, since $U_{\text{right}} \leq \Delta_1$ and $(G/L)_{\text{right}} \leq \Delta_0$, it suffices to verify that

$$\Delta_1 \approx_2 \text{Aut}(\mathcal{A}_U) \quad \text{and} \quad \Delta_0 \approx_2 \text{Aut}(\mathcal{A}_{G/L}).$$

In its turn, to prove these relations it suffices to verify by Corollary A5.7 applied to the S-ring \mathcal{A}_U and the groups Γ_4, Γ_3 , and the S-ring $\mathcal{A}_{G/L}$ and the groups Γ_6, Γ_5 , that

$$\Gamma_4 \approx_2 \text{Aut}(\mathcal{A}_{H_2}), \quad \Gamma_3 \approx_2 \text{Aut}(\mathcal{A}_{U/K}), \quad \Gamma_6 \approx_2 \text{Aut}(\mathcal{A}_{V/L}), \quad \Gamma_5 \approx_2 \text{Aut}(\mathcal{A}_{G/H_1}).$$

However, the statements on Γ_3 and Γ_6 hold by the definition of these groups. Next, the hypothesis $\mathcal{A}_K = \mathbb{Z}K$ implies by Theorem 7.1 that \mathcal{A}_{H_2} is the tensor product of the cyclotomic rings \mathcal{A}_K and $\mathcal{A}_N \cong \mathcal{A}_{H_2/K}$ (the latter S-ring is cyclotomic by Theorem 3.1). Therefore the S-ring \mathcal{A}_{H_2} is cyclotomic, and hence the groups $\text{Aut}(\mathcal{A}_{H_2})$ and Γ_4 are 2-equivalent. Similarly, one can prove that the group Γ_5 is 2-equivalent to the group $\text{Aut}(\mathcal{A}_{G/H_1})$.

To prove (25) we note that due to (26), (27) and (28) we have

$$(\Gamma_3)^{U/H_1} = \Gamma_1 = (\Gamma_5)^{U/H_1} \quad \text{and} \quad (\Gamma_4)^{H_2/L} = \Gamma_2 = (\Gamma_6)^{H_2/L}. \quad (29)$$

Therefore both $(\Delta_1)^{U/L}$ and $(\Delta_0)^{U/L}$ are contained in the group Δ . To prove the converse inclusion we observe that due to (24) there is an isomorphism

$$G/V \rightarrow U/H_2, \quad X \mapsto X \cap U =: Y. \quad (30)$$

In what follows the factor sets X and Y modulo L are denoted by \overline{X} and \overline{Y} respectively. For each $X \in G/V$ we fix a bijection $h_X \in (G_{right})^{V,X}$ that takes H_2 to Y , and set

$$h_Y = (h_X)^Y, \quad h_{\overline{X}} = (h_X)^{\overline{X}}, \quad h_{\overline{Y}} = (h_X)^{\overline{Y}}. \quad (31)$$

Then due to equality (22) any element $\overline{f} \in \Delta$ can be written in the form

$$\overline{f} = \{f_{\overline{Y}}\} \wr_{H_2/H_1} \overline{f}_0$$

for some permutation $\overline{f}_0 \in \Gamma_1$ and a family of permutations $f_{\overline{Y}} \in \Gamma_2$ where $Y \in U/H_2$, such that

$$(f_{\overline{Y}})^{H_2/H_1} = (h_{\overline{Y}})^{H_2/H_1} (\overline{f}_0)^{Y/H_1} ((h_{\overline{Y}'})^{H_2/H_1})^{-1} \quad (32)$$

for all $Y \in U/H_2$ where Y' is the H_2 -coset in U for which $Y'/H_1 = (Y/H_1)^{\overline{f}_0}$. In what follows we find some elements of the groups Δ_1 and Δ_0 the restrictions of which to U/L coincide with \overline{f} .

To find the required permutation in Δ_0 we observe that $\mathcal{A}_{M/H_1} = \mathbb{Z}M/H_1$ because $\mathcal{A}_{M/H_1} \cong \mathcal{A}_{G/U}$, and the latter is a group ring by condition (2). So by Theorem 7.1 we have

$$\Gamma_5 = (M/H_1)_{right} \otimes \Gamma_1. \quad (33)$$

Therefore this group contains the permutation $f_0 = \text{id}_{M/H_1} \otimes \overline{f}_0$. Clearly,

$$(f_0)^{U/H_1} = \overline{f}_0. \quad (34)$$

Next, let $X \in G/V$. Then by (27) there exists a permutation $f_{\overline{X}} \in \Gamma_6$ that leaves the set H_2/L fixed and such that

$$(f_{\overline{X}})^{H_2/L} = f_{\overline{Y}}. \quad (35)$$

Below we show that

$$(f_{\overline{X}})^{V/H_1} = (h_{\overline{X}})^{V/H_1} (f_0)^{X/H_1} ((h_{\overline{X}'})^{V/H_1})^{-1} \quad (36)$$

where X' is the V -coset in G for which $X'/H_1 = (X/H_1)^{f_0}$. Then one can define a permutation $f = \{f_{\overline{X}}\} \wr_{V/H_1} f_0$ belonging to the group Δ_0 . This is we wanted to find because $f^{U/L} = \overline{f}$ by (31), (34) and (35).

To prove (36) we observe that $(\Gamma_6)^{V/H_1} = (V/H_2)_{right} \otimes \text{Hol}_{\mathcal{A}}(V/M)$ because $\mathcal{A}_{V/H_2} = \mathbb{Z}V/H_2$ (see above). So

$$(f_{\overline{X}})^{V/H_1} = (f_{\overline{X}})^{V/H_2} \otimes (f_{\overline{X}})^{V/M} = (f_{\overline{X}})^{V/H_2} \otimes (f_{\overline{Y}})^{V/M}.$$

On the other hand, the permutation $f_{\overline{X}}$ leaves the set H_2/L fixed. So $(f_{\overline{X}})^{V/H_2}$ lives the set H_2 fixed. Since also $(f_{\overline{X}})^{V/H_2} \in (V/H_2)_{right}$, this implies that $(f_{\overline{X}})^{V/H_2} = \text{id}_{V/H_2}$. Thus (36) holds by (32) and the choice of the bijections h_X .

To find a permutation $f \in \Delta_1$ such that $f^{U/L}$ coincides with the permutation \overline{f} defined in (30), we observe that due to (26) there exists a permutation $f_0 \in \Gamma_3$ such that equality (34) holds. Next, for each $Y \in U/H_2$ we define a permutation of H_2/K defined by

$$g_Y = (h_Y)^{H_2/K} (f_0)^{Y/K} ((h_{Y'})^{H_2/K})^{-1} \quad (37)$$

where Y' is the H_2 -coset in U for which $Y'/K = (Y/K)^{f_0}$. However, the bijection h_X by its choice leaves the set U fixed. So

$$(h_Y)^{H_2/K} (f_0)^{Y/K} ((h_{Y'})^{H_2/K})^{-1} = ((h_X)^{U/K} f_0 ((h_{X'})^{U/K})^{-1})^{H_2/K}$$

Thus g_Y belongs to the group $(\Gamma_3)^{H_2/K} = (\Gamma_4)^{H_2/K}$. Therefore, due to condition (2) of the theorem the permutation

$$f_Y := g_Y \otimes (f_{\overline{Y}})^{H_2/N}$$

belongs to the group Γ_4 . Moreover, equalities (34), (37) and (32) imply that

$$(f_Y)^{H_2/L} = (h_Y)^{H_2/L} (f_0)^{Y/L} (h_{Y'})^{H_2/L})^{-1}.$$

Thus one can define a permutation $f = \{f_Y\} \wr_{H_2/L} f_0$ belonging to the group Δ_1 . By the choice of f_0 and f_Y we have $f^{U/L} = \overline{f}$, which completes the proof. ■

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